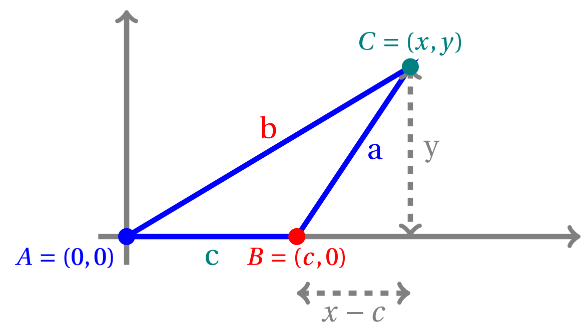
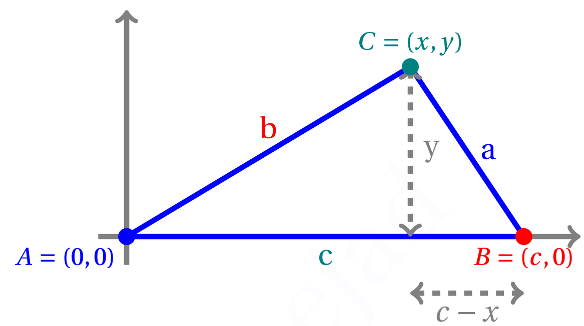
8.2: Non-right Angle Triangles: Law of Cosines

- **Derivation of the formula:** Use either picture.



OR



In the right triangle, we see that $a^2 = y^2 + (x-c)^2$. (Note that $(x-c)^2 = (c-x)^2$.)

On the other hand, $y = b\sin(\angle A)$ and $x = b\cos(\angle A)$

$a^2 = (b\cos(\angle A) - c)^2 + (b\sin(\angle A) - 0)^2$.

Expand and $a^2 = b^2\cos^2(\angle A) - 2bc\cos(\angle A) + c^2 + b^2\sin^2(\angle A)$.

$a^2 = b^2\left(\cos^2(\angle A) + \sin^2(\angle A)\right) + c^2 - 2bc\cos(\angle A) \implies a^2 = b^2 + c^2 - 2bc\cos(\angle A)$

- **Why Law of Cosines?** Even though the Law of Sines is a tool for calculating triangles with known values ASA, AAS and SSA, we need another tool for triangles with known values which are SSS and SAS.

- **How to calculate the triangles?**

  SSS: First use the Law of Cosines once to find the angle opposite of the longest side. Then you can use the Law of Sines or the Law of Cosines to calculate the rest.

  SAS: First use the law of Cosines to find the third side. Then use either of the laws to find the rest of the values. In the second step, either find the angle opposite of the longest side using the law of cosines or find the smaller unsolved angle opposite of the shorter side using Law of Sines to avoid double solutions by the law of sines.

- **The Law of Cosines**

$a^2 = b^2 + c^2 - 2bc\cos(\angle A)$ $\qquad$ $b^2 = a^2 + c^2 - 2ac\cos(\angle B)$ $\qquad$ $c^2 = a^2 + b^2 - 2ab\cos(\angle C)$

- **Heron's formula:**

The area of a triangle with sides a, b and c is Area $= \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$ is the semi-perimeter of the triangle.

Worked out Examples:

1. For a triangle with $a = 13$, $b = 8$ and $c = 15$, use the Law of Cosines to find $\angle B$.

Solution:

In this problem, we are solving for only one angle. There is no need to make a decision on solving for the largest or the smallest angle; since the problem is giving the direction.

Use $b^2 = a^2 + c^2 - 2ac\cos(\angle B) \implies 8^2 = 13^2 + 15^2 - 2(13)(15)\cos(\angle B)$

$\underset{\text{Solve}}{\implies} \cos(\angle B) = 0.85 \implies \angle B = \arccos(0.85) \approx \boxed{32.2^\circ}$

2. Use the Law of Sines to find all possible triangles with $\angle B = 26^\circ$, $b = 9$ and $c = 12$.

Solution:

$$\frac{\sin(26^\circ)}{9} = \frac{\sin(\angle C)}{12} \implies \sin(\angle C) = \frac{12\sin 26^\circ}{9} \approx \boxed{.584}$$

First possibility: $\angle C = \arcsin(.584) = \boxed{35.7^\circ}$ or The second possibility: $\angle C = 180^\circ - 35.7^\circ = \boxed{144.2^\circ}$

First triangle : $\angle C = \boxed{35.7^\circ} \implies \angle A = 180^\circ - 35.7^\circ - 26^\circ = \boxed{118.3^\circ}$

Now use the Law of Sines again: $\dfrac{\sin(26^\circ)}{9} = \dfrac{\sin(118.3^\circ)}{a}$ So $a = \dfrac{9\sin(118.3^\circ)}{\sin(26^\circ)} \approx \boxed{18}$

Second triangle: $\angle C = \boxed{144.3^\circ} \implies \angle A = 180^\circ - 144.3^\circ - 26^\circ = \boxed{9.7^\circ}$

Now use the Law of Sines again: $\dfrac{\sin(26^\circ)}{9} = \dfrac{\sin(9.8^\circ)}{a} \implies a = \dfrac{9\sin(9.8^\circ)}{\sin(26^\circ)} \approx \boxed{3.5}$

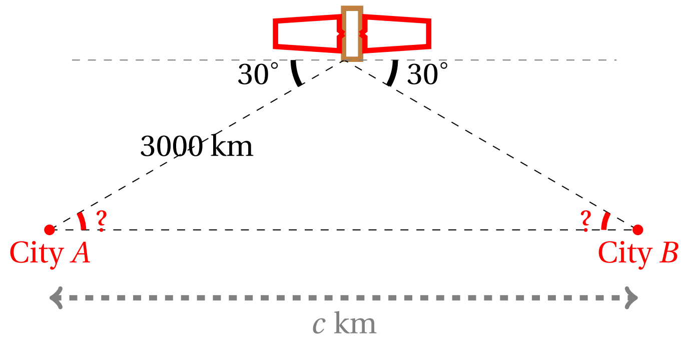
Triangle one	Triangle two
$\angle B = 26^\circ$	$\angle B = 26^\circ$
$b = 9$	$b = 9$
$c = 12$	$c = 12$
$\angle C = 35.7^\circ$	$\angle C = 144.3^\circ$
$\angle A = 118.3^\circ$	$\angle A = 9.7^\circ$
$a = 18$	$a = 3.5$

1. Use the Law of Cosines to determine side c and angles $\angle A$ and $\angle B$ if $a = 23$ and $b = 34$ and $\angle C = 56^{\circ}$.

2. Find all angles of a triangle with sides 6 , 4 and 4.

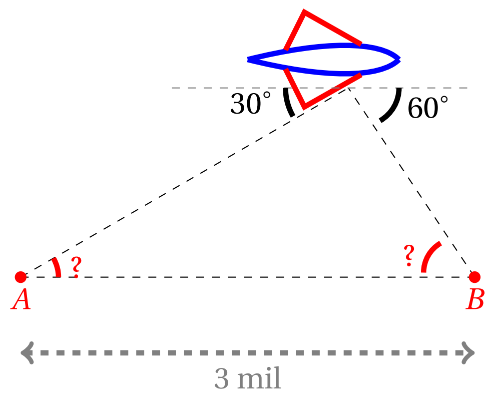
3. Use Heron's Formula, to find the area of a triangle with sides: $a = 6$ and $b = 9$ and $c = 5$.

4. **Astronomy and Aerospace Engineering:** A satellite calculates the distances and angles as shown below. Find the distance between the two cities.



5. Use the Law of Sines to find all possible triangles with $\angle B = 60^\circ$, $b = 7$ and $c = 8$.

6. Yanru is flying her Su 2018 plane over a straight highway. She determines the angles of depression to two mileposts, 3 mil apart, to be $\angle x = 30^\circ$ and $\angle y = 60^\circ$, as shown in the figure.



(a) Find the distance of the plane from point A.

(b) Find the elevation of the plane.

7. While driving through the flat lands of Western Kansas, Yi notices the Rocky Mountains directly in front of him. The angle of elevation to the peak of the mountains is 5.1°. After Yi drives 20 miles closer to the mountain, the angle of elevation is 11°. Use Yi's measurements to approximate the **height of the mountain** from the road.